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Hopf Modules and Yetter–Drinfel'd Modules

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1. INTRODUCTION

Yetter–Drinfel'd modules over a bialgebra H were introduced by Yetter in [11] (under the name of crossed bimodules; they are called Yang–Baxter modules in [2] and Yetter–Drinfel'd structures in [7]). A (say, right) Yetter–Drinfel'd module is a right H-module and a right H-comodule, the action and coaction satisfying a compatibility condition. The main feature of the definition is that Yetter–Drinfel'd modules form a pre-braided monoidal category. Under favourable conditions (e.g., if H is a Hopf algebra with bijective antipode, see also the detailed discussion and many examples in [2]), the category is even braided (or quasisymmetric). Via the (pre-)braiding structures the notion of Yetter–Drinfel'd module plays a part in the relations between knot theory and quantum groups.

A (say, left) Hopf module is also a (left) H-module and a (left) H-comodule, the action and coaction satisfying a very different compatibility condition. The definition is best understood by a look at the structure theorem on Hopf modules (see [8, p. 84]): Every left Hopf module is isomorphic to one of the form $H \otimes V$, where V is a vector space and $H \otimes V$ has the structures induced by those of H. Thus we have an equivalence between the category of Hopf modules and the category of vector spaces. Hopf modules (and the structure theorem) were reinvented by Woronowicz in [10] to study differential calculi over quantum groups. The object replacing the module of differential one-forms of a Lie group in [10] is a two-sided H-module and a two-sided H-comodule satisfying the Hopf module compatibility between each of the module and each of the comodule structures. We shall call this structure (a bicovariant bimodule in the terminology of [10]) a two-sided two-cosided Hopf module. Woronowicz shows that every Hopf module is a free H-module (the structure theorem from [8]) and classifies two-sided two-cosided Hopf modules by additional structures given in terms of the chosen basis.

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Our main theorem (5.7) states that the structure theorem on Hopf modules extends to a category equivalence between two-sided two-cosided Hopf modules over a Hopf algebra H, and the category of right Yetter–Drinfel'd structures. The equivalence is even a monoidal one if we endow the category of Yetter–Drinfel'd structures with the tensor product over k with the diagonal action and codiagonal coaction (see [11, 2, 7]), and the category of two-sided two-cosided Hopf modules with either the tensor product or the cotensor product over H. Our category equivalence can be seen as a coordinate free version of the classification in [10].

2. Preliminaries

Let $\mathscr{C}$ denote a symmetric monoidal category with unit object k. For simplicity (actually without loss of generality by the coherence theorems in [3]) we shall assume the tensor product $\otimes$ to be strictly associative, $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, and $k \otimes X = X = X \otimes k$. By algebras, coalgebras, Hopf algebras, modules, and comodules we shall always mean algebras, etc., in $\mathscr{C}$, unless stated otherwise. We denote ∇: $A \otimes A \to A$ and η: $k \to A$ the multiplication and unit morphisms of an algebra A, μ: $A \otimes M \to M$ the structure morphism of a (left) A-module M, Δ: $C \to C \otimes C$ and ε: $C \to k$ the comultiplication and counit of a coalgebra C, and δ: $M \to M \otimes C$ the structure morphism of a (right) C-comodule M. We denote the antipode of a Hopf algebra by S: $H \to H$. Throughout the paper, we shall make free use of the notion of generalized elements of objects in $\mathscr{C}$. See [4, 5] for a reference on algebras, modules, coalgebras, etc., in monoidal categories including a discussion of generalized elements in this context. In particular, we use the notations $\Delta(c) = c_{(1)} \otimes c_{(2)}$ and $\delta(m) = m_{(0)} \otimes m_{(1)}$ for generalized elements c of a coalgebra C and m of a right C-comodule M; we use the notation $\delta(m) = m_{(-1)} \otimes m_{(0)}$ for left comodules.

The tensor product over an algebra A in $\mathscr{C}$ of a right A-module M and a left A-module N is defined to be the coequalizer

$$M \otimes A \otimes N \rightrightarrows M \otimes N \to M \otimes_A N$$

if it exists. Note that if N has the form $A \otimes V$ with the structure induced by that of A, then the coequalizer exists and splits. Stated differently, we have the familiar isomorphism

$$\begin{aligned} \xi: M \otimes V &\cong M \otimes_A (A \otimes V) \\ m \otimes v &\mapsto m \otimes 1 \otimes v \\ ma \otimes v &\leftarrow m \otimes a \otimes v. \end{aligned}$$

If M, N are A–A-bimodules and $\otimes$ preserves the coequalizer, then $M \otimes_A N$ is also an A–A-bimodule.

The cotensor product over a coalgebra C of a right C-comodule M and a left C-comodule N is defined to be their tensor product in the dual category $\mathscr{C}^0$ (over the algebra C); that is, it is an equalizer

$$M \,\square_C\, N \to M \otimes N \rightrightarrows M \otimes C \otimes N,$$

if it exists. The equalizer exists and is split if M has the form $C \otimes V$. If M,N are two-sided comodules and $\otimes$ preserves the equalizer than $M \,\square_C\, N$ is also a two-sided comodule.

3. Hopf Modules

Let H be a bialgebra. We denote by $\mathscr{M}^H$ the category of all right H-comodules, by ${}_H\mathscr{M}$ the category of all left H-modules, by ${}_H\mathscr{M}_H$ the category of H–H-bimodules, etc. It is well known that these categories are monoidal.

3.1. Definition and Lemma. *Let $M \in \mathscr{C}$. The following statements are equivalent*:

(1) *M is a right H-module in $\mathscr{M}^H$.*

(2) *M is a right H-comodule in $\mathscr{M}_H$.*

(3) *M is a right H-module and a right H-comodule such that for all $h \in H$ and $m \in M$ we have $\delta(mh) = m_{(0)}h_{(1)} \otimes m_{(1)}h_{(2)}$.*

If these conditions are satisfied, M is said to be a (right) Hopf module. We denote by $\mathscr{M}_H^H$ the category of all right Hopf modules. In the same manner, we define the categories ${}^H\mathscr{M}_H$, ${}_H\mathscr{M}^H$, and ${}_H^H\mathscr{M}$.

Thus the notion of a Hopf module is self-dual (it stays the same when we replace $\mathscr{C}$ by $\mathscr{C}^0$, the dual category).

3.2. Definition and Lemma. *Let $M \in \mathscr{C}$. The following statements are equivalent*:

(1) *M is an H–H-bimodule in $\mathscr{M}^H$.*

(2) *M is a right H-comodule in ${}_H\mathscr{M}_H$.*

(3) *M is an H–H-bimodule and a right H-comodule such that $M \in {}_H\mathscr{M}^H$ and $M \in \mathscr{M}_H^H$.*

We call these objects two-sided Hopf modules and denote the category of all these (with those morphisms that are left and right linear and right colinear) by ${}_H\mathscr{M}_H^H$. The category ${}_H^H\mathscr{M}_H$ is defined in the same manner.

3.3. *Remark.* If $M, N \in {}_H\mathcal{M}_H^H$, then $M \otimes N$ with the codiagonal comodule structure, the left module structure induced by that of M, and the right module structure induced by that of N is also a two-sided Hopf module. (This is just the tensor product of a left module and a right module in the category of comodules.) The tensor product $M \otimes_H N$ over H is a quotient of this two-sided Hopf module if the tensor product exists in $\mathscr{C}$ and is preserved by $\otimes$: It is a quotient of $M \otimes N$ as a comodule since the underlying functor $\mathcal{M}^H \to \mathscr{C}$ creates colimits by [4, Cor. 2.4], and thus it is just the tensor product over H of the H–H-bimodules M and N in $\mathcal{M}^H$. In particular, if all the coequalizers defining tensor products of two-sided Hopf modules exist in $\mathscr{C}$ and are preserved under $\otimes$ (e.g., if $\mathscr{C}$ is the category of modules over a commutative ring k), then ${}_H\mathcal{M}_H^H$ is a monoidal category with tensor product $\otimes_H$.

3.4. Definition. A two-cosided Hopf module is a two-sided Hopf module in the dual category $\mathscr{C}^0$.

3.5. *Remark.* Dually to the remark above, if $M, N \in {}^H\mathcal{M}_H^H$, then $M \otimes N$ with the diagonal module structure, the left comodule structure induced by that of M, and the right comodule structure induced by that of N is also a two-cosided Hopf module. The cotensor product $M \square_H N$ is a subobject of this two-cosided Hopf module if it exists and is preserved by $\otimes$. If all the equalizers defining cotensor products of two-cosided Hopf modules exist in $\mathscr{C}$ and are preserved by $\otimes$ (e.g., if $\mathscr{C}$ is the category of vector spaces over a field (!) k), then $({}^H\mathcal{M}_H^H, \square_H)$ is a monoidal category.

3.6. Definition and Lemma. *The following statements are equivalent*:

(1) *M is a two-sided H-module in the category ${}^H\mathcal{M}^H$.*

(2) *M is a two-sided H-comodule in the category ${}_H\mathcal{M}_H$.*

(3) *M is a two-sided H-module and a two-sided H-comodule such that $M \in \mathcal{M}_H^H, {}_H\mathcal{M}^H, {}^H\mathcal{M}_H, {}^H_H\mathcal{M}$.*

We then call M a two-sided two-cosided Hopf module, and we denote the category of these objects with the morphisms that are linear and colinear on both sides by ${}^H_H\mathcal{M}_H^H$.

Note that the category of two-sided two-cosided Hopf modules has (under suitable assumptions on equalizers and coequalizers, for example if $\mathscr{C}$ is the category of vector spaces over a field k) at least two structures of a monoidal category, the tensor product of $M, N \in {}^H_H\mathcal{M}_H^H$ being defined by $M \otimes_H N$ or $M \square_H N$ with the structures as discussed in 3.3 and 3.5. The two monoidal categories are of course that of H–H-bimodules in ${}^H\mathcal{M}^H$

with the tensor product over H, and that of H–H-bicomodules in ${}_H\mathscr{M}_H$ with the cotensor product over H, respectively.

4. Yetter–Drinfel'd Modules

A Yetter–Drinfel'd module is a right H-module and a right H-comodule with a compatibility condition very different from the one defining a Hopf module.

4.1. Definition. A right–right Yetter–Drinfel'd module $(V, \leftharpoonup, \delta)$ is a right H-module and a right H-comodule such that for all $h \in H$ and $v \in V$,

$$v_{(0)} \leftharpoonup h_{(1)} \otimes v_{(1)}h_{(2)} = (v \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(v \leftharpoonup h_{(2)})_{(1)},$$

we denote by $\mathscr{Y}\mathscr{D}_H^H$ the category of right Yetter–Drinfel'd modules and the morphisms that are both linear and colinear.

The categories ${}_H\mathscr{Y}\mathscr{D}^H$, ${}^H\mathscr{Y}\mathscr{D}_H$, and ${}_H^H\mathscr{Y}\mathscr{D}$ of left–right, right–left, and left–left Yetter–Drinfel'd modules, respectively, are defined by the compatibility conditions

$$\begin{aligned} h_{(1)} \leftrightarrows v_{(0)} \otimes h_{(2)}v_{(1)} &= (h_{(2)} \leftrightarrows v)_{(0)} \otimes (h_{(2)} \leftrightarrows v)_{(1)}h_{(1)} \\ v_{(-1)}h_{(1)} \otimes v_{(0)} \rightleftharpoons h_{(2)} &= h_{(2)}(v \rightleftharpoons h_{(1)})_{(-1)} \otimes (v \rightleftharpoons h_{(1)})_{(0)} \\ h_{(1)}v_{(-1)} \otimes h_{(2)} \rightharpoonup v_{(0)} &= (h_{(1)} \rightharpoonup v)_{(-1)}h_{(2)} \otimes (h_{(1)} \rightharpoonup v)_{(0)}. \end{aligned}$$

Yetter–Drinfel'd modules have been studied in relation to solutions of the quantum Yang–Baxter equation and low dimensional topology. We cite only [11, 6, 2, 7]. The most important property (see [11, 7]) is

4.2. Theorem. *$\mathscr{Y}\mathscr{D}_H^H$ is a pre-braided monoidal category. The tensor product of $V, W \in \mathscr{Y}\mathscr{D}_H^H$ is $V \otimes W$ with the codiagonal comodule and the diagonal module structure. The pre-braiding is given by*

$$\begin{aligned} \sigma_{VW} \colon V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w_{(0)} \otimes v \leftharpoonup w_{(1)}. \end{aligned}$$

It is a braiding if H is a Hopf algebra with bijective antipode.

In the rest of this section we want to show that Yetter–Drinfel'd modules actually occur in [10].

Let k be a field, H a Hopf algebra over k and V a finite dimensional vector space. Let $(v_1, \dots, v_n)$ be a basis of V. In what follows, indices always run through $\{1, \ldots, n\}$ and summation is implied whenever an index appears twice, once as an upper and once as a lower index.

Then right H-comodule structures on V are in bijection with multiplicative matrices in H, that is, matrices R_i^j of elements of H satisfying $\Delta(R_i^j) = R_k^j \otimes R_i^k$, via the formula $\delta(v_i) = v_j \otimes R_i^j$.

Right H-module structures on V are in bijection with algebra maps $H \to \mathrm{End}(V)$, that is, with matrices f_i^j of linear functionals on H satisfying $f_i^j(gh) = f_i^k(g) f_k^j(h)$, via the formula $v_i \leftharpoonup h = v_j f_i^j(h)$.

Write $f * h := h_{(1)} f(h_{(2)})$ and $h * f = f(h_{(1)}) h_{(2)}$ for $f \in H^*$ and $h \in H$. We claim that V is a Yetter–Drinfel'd module iff

$$R_i^j \cdot \left(h * f_j^k\right) = \left(f_i^j * h\right) \cdot R_j^k$$

holds for all $h \in H$ and indices i, k. Compare [10, Eq. (2.39)]. Indeed, for $h \in H$ we have

$$\begin{aligned} v_{i(0)} \leftharpoonup h_{(1)} \otimes v_{i(1)} h_{(2)} &= v_j \leftharpoonup h_{(1)} \otimes R_i^j h_{(2)} \\ &= v_k f_j^k\left(h_{(1)}\right) \otimes R_i^j h_{(2)} \\ &= v_k \otimes R_i^j \cdot \left(h * f_j^k\right) \end{aligned}$$

and

$$\begin{aligned} \left(v_i \leftharpoonup h_{(2)}\right)_{(0)} \otimes h_{(1)}\left(v_i \leftharpoonup h_{(2)}\right)_{(1)} &= v_{j(0)} f_i^j\left(h_{(2)}\right) \otimes h_{(1)} v_{j(1)} \\ &= v_k \otimes \left(f_i^j * h\right) \cdot R_j^k \end{aligned}$$

so that our claim follows from the fact that (v_i) is a basis of V.

5. A Category Equivalence

5.1. Theorem. *Let V be an object of $\mathscr{C}$, and let $H \otimes V \in {}_H^H\mathscr{M}$ with the structures induced by those of H. There is a bijection between*

(1) *Right H-module structures on $H \otimes V$ making $H \otimes V$ an object of ${}_H^H\mathscr{M}_H$.*

(2) *Right H-module structures on V.*

More precisely, if V is a right H-module then the diagonal module structure on $H \otimes V$ makes $H \otimes V$ a two-sided Hopf module. Conversely, if $H \otimes V$ is a two-sided Hopf module, then there is a unique right module structure on V

making $\varepsilon \otimes V$: $H \otimes V \to V$ a morphism of right modules, namely

$$V \otimes H \xrightarrow{\eta \otimes V \otimes H} H \otimes V \otimes H \xrightarrow{\mu} H \otimes V \xrightarrow{\varepsilon \otimes V} V,$$

that is, $v \leftharpoonup h = (\varepsilon \otimes V)((1 \otimes v)h)$.

To prove this, we take a detour via a theorem describing right module structures on free left modules in terms of "commutation relations" (see [9, Prop. 2.1]).

5.2. THEOREM. *Let A and B be algebras and V an object of $\mathscr{C}$. Endow $A \otimes V$ with the obvious structure of a left A-module. There is a bijection between*

(1) *right B-module structures making $A \otimes V$ an A–B bimodule;*

(2) *morphisms f: $V \otimes B \to A \otimes V$ in $\mathscr{C}$, making the following diagrams commute:*

$$\begin{tikzcd}
V \otimes B \otimes B \arrow[r, "f \otimes B"] \arrow[d, "V \otimes \nabla"'] & A \otimes V \otimes B \arrow[r, "A \otimes f"] & A \otimes A \otimes V \arrow[d, "\nabla \otimes V"] \\
V \otimes B \arrow[rr, "f"'] & & A \otimes V
\end{tikzcd}$$

$$\begin{tikzcd}
& V \arrow[dl, "V \otimes \eta"'] \arrow[dr, "\eta \otimes V"] & \\
V \otimes B \arrow[rr, "f"'] & & A \otimes V
\end{tikzcd}$$

Proof. Assume first that we are given a right B-module structure μ: $A \otimes V \otimes B \to A \otimes V$ making $A \otimes V$ an A–B-bimodule, where the left module structure is of course $\nabla \otimes V$: $A \otimes A \otimes V \to A \otimes V$. Put $f = \mu(\eta_A \otimes V \otimes B)$. Then

$$\begin{aligned}
&(\nabla_A \otimes V)(A \otimes f)(f \otimes B) \\
&\quad = (\nabla_A \otimes V)(A \otimes \mu)(A \otimes \eta_A \otimes V \otimes B)(\mu \otimes B)(\eta_A \otimes V \otimes B \otimes B) \\
&\quad = \mu(\nabla_A \otimes V \otimes B)(A \otimes \eta_A \otimes V \otimes B)(\mu \otimes B)(\eta_A \otimes V \otimes B \otimes B) \\
&\quad = \mu(\mu \otimes B)(\eta_A \otimes V \otimes B \otimes B) \\
&\quad = \mu(A \otimes V \otimes \nabla_B)(\eta_A \otimes V \otimes B \otimes B) \\
&\quad = \mu(\eta_A \otimes V \otimes B)(V \otimes \nabla_B) \\
&\quad = f(V \otimes \nabla_B),
\end{aligned}$$

where we used left linearity of μ, that is, $(\nabla_A \otimes V)(A \otimes \mu) = \mu(\nabla_A \otimes V \otimes$

B), in the second equality, and associativity of μ in the fourth. Commutativity of the second diagram is derived from the axiom of unit for μ.

Now assume f is given making the two diagrams commute. Define μ: $A \otimes V \otimes B \xrightarrow{A \otimes f} A \otimes A \otimes V \xrightarrow{\nabla_A \otimes V} A \otimes V$. Then by definition μ is left A linear. μ is associative since

$$\begin{aligned}
\mu(\mu \otimes B) &= (\nabla_A \otimes V)(A \otimes f)(\nabla_A \otimes V \otimes B)(A \otimes f \otimes B) \\
&= (\nabla_A \otimes V)(\nabla_A \otimes A \otimes V)(A \otimes A \otimes f)(A \otimes f \otimes B) \\
&= (\nabla_A \otimes V)(A \otimes \nabla_A \otimes V)(A \otimes A \otimes f)(A \otimes f \otimes B) \\
&= (\nabla_A \otimes V)(A \otimes f)(A \otimes V \otimes \nabla_B) \\
&= \mu(A \otimes V \otimes \nabla_B).
\end{aligned}$$

The equation $\mu(A \otimes V \otimes \eta) = \mathrm{id}_{A \otimes V}$ is derived from the second diagram. ∎

Proof of 5.1. Applying the preceding theorem to the left H-module $H \otimes V$ in the category of left H-comodules, we get an equivalence between the data in (1) and colinear maps f: $V \otimes H \to H \otimes V$ satisfying

$$f(V \otimes \nabla) = (\nabla \otimes V)(H \otimes f)(f \otimes H) \qquad (1)$$

$$f(V \otimes \eta) = \eta \otimes V. \qquad (2)$$

Now for any left H-comodule W the mapping

$$E_W\colon \mathrm{Hom}^H(W, H \otimes V) \ni f \mapsto (\varepsilon \otimes V)f \in \mathrm{Hom}(W, V)$$

is a bijection. Let f: $V \otimes H \to H \otimes V$ and $\rho = (\varepsilon \otimes V)f$. We have to show that f satisfies the equations above iff ρ defines a right module structure on V. We have

$$\begin{aligned}
(\varepsilon \otimes V)(\nabla \otimes V)(H \otimes f)(f \otimes H) &= (\varepsilon \otimes \varepsilon \otimes V)(H \otimes f)(f \otimes H) \\
&= (\varepsilon \otimes H)(H \otimes \rho)(f \otimes H) \\
&= \rho(\rho \otimes H)
\end{aligned}$$

and $(\varepsilon \otimes V)f(V \otimes \nabla) = \rho(V \otimes \nabla)$, so that ρ satisfies the associativity axiom iff $E_{V \otimes H \otimes H}((\nabla \otimes V)(H \otimes f)(f \otimes H)) = E_{V \otimes H \otimes H}(f(V \otimes \nabla))$, which, since $E_{V \otimes H \otimes H}$ is a bijection, is equivalent to (1). The equivalence between (2) and the axiom of unit for ρ is treated similarly. ∎

The proof of 5.1 was given in the framework of general monoidal categories. Applying the theorem to the opposite category gives:

5.3. Corollary. *With hypotheses as in* 5.1 *there is a bijection between right comodule structures on* $H \otimes V$ *making* $H \otimes V$ *a two-cosided Hopf* *module, and right H-comodule structures on V. For a given comodule structure on V, the corresponding comodule structure on $H \otimes V$ is the codiagonal structure. The structure on V is given by the composition*

$$V \xrightarrow{\eta \otimes V} H \otimes V \xrightarrow{\delta} H \otimes V \otimes H \xrightarrow{\varepsilon \otimes H} V \otimes H.$$

It is the unique one making $\eta \otimes V \colon V \to H \otimes V$ a colinear map.

5.4. THEOREM. *Let $V \in \mathscr{C}$, and endow $H \otimes V$ with the obvious structure of a left Hopf module. Then there is a bijection between the following data*:

(1) *a right H-module structure and a right H-comodule structure on $H \otimes V$ making $H \otimes V$ a two-sided two-cosided H Hopf module,*

(2) *a structure of right-right Yetter–Drinfel'd module on V,*

which is induced by the bijections in 5.1 *and* 5.3.

Proof. All that remains to be shown is that the condition on the module and comodule structures on V that they define a Yetter–Drinfel'd module is equivalent to the condition that $H \otimes V$ be a right Hopf module. We give a proof using generalized elements. If $(V, \leftharpoonup)$ is a right module and (V, δ) is a right comodule, then (denoting δ_{r} the induced right comodule structure on $H \otimes V$) we have for $g, h \in H$ and $v \in V$:

$$\begin{aligned}
\delta_{\mathrm{r}}((g \otimes v)h) &= \delta_{\mathrm{r}}(gh_{(1)} \otimes v \leftharpoonup h_{(2)}) \\
&= g_{(1)}h_{(1)} \otimes (v \leftharpoonup h_{(3)})_{(0)} \otimes g_{(2)}h_{(2)}(v \leftharpoonup h_{(3)})_{(1)} \\
\delta_{\mathrm{r}}(g \otimes v)\Delta(h) &= (g_{(1)} \otimes v_{(0)})h_{(1)} \otimes g_{(2)}v_{(1)}h_{(2)} \\
&= g_{(1)}h_{(1)} \otimes v_{(0)} \leftharpoonup h_{(2)} \otimes g_{(2)}v_{(1)}h_{(3)}.
\end{aligned}$$

Thus on one hand the condition that $H \otimes V$ be a right Hopf module, that is, $\delta((g \otimes v)h) = \delta(g \otimes v)\Delta(h)$, follows from the condition that V be a right Yetter–Drinfel'd module. On the other hand, applying $\varepsilon \otimes V \otimes H$ to the equation $\delta((1 \otimes v)h) = \delta(1 \otimes v)\Delta(h)$ yields

$$\begin{aligned}
&(v \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(v \leftharpoonup h_{(2)})_{(1)} \\
&\quad = (\varepsilon \otimes V \otimes H)\big(h_{(1)} \otimes (v \leftharpoonup h_{(3)})_{(0)} \otimes h_{(2)}(v \leftharpoonup h_{(3)})_{(1)}\big) \\
&\quad = (\varepsilon \otimes V \otimes H)(\delta((1 \otimes v)h)) \\
&\quad = (\varepsilon \otimes V \otimes H)(\delta(1 \otimes v)\Delta(h)) \\
&\quad = (\varepsilon \otimes V \otimes H)\big(h_{(1)} \otimes v_{(0)} \leftharpoonup h_{(2)} \otimes v_{(1)}h_{(3)}\big) \\
&\quad = v_{(0)} \leftharpoonup h_{(1)} \otimes v_{(1)}h_{(2)}
\end{aligned}$$

and thus V is a Yetter–Drinfel'd module if $H \otimes V$ is a right Hopf module. ∎

From now on we shall always assume that $\mathscr{C}$ has equalizers (we could as well assume it has coequalizers). It is a well-known theorem (see [8, p. 84]) that, for k a field, all Hopf modules over a k-Hopf algebra H actually have the form $H \otimes V$ for a k-vector space V. The analogous result, stated below, holds in a symmetric monoidal category with equalizers; we will not give a detailed proof. The referee has pointed out Ref. [1], where the authors develop a theory of relative Hopf modules over comodule algebras in a symmetric monoidal category. In the case in which $H \otimes (-)$ preserves equalizers and reflects isomorphisms, 5.5 is just a special case of [1, Cor. (2.6)].

For a left Hopf module M we denote by ${}^{\mathrm{co}\,H}M$ the equalizer of $\delta, \eta \otimes M \colon M \to H \otimes M$. Thus, ${}^{\mathrm{co}\,H}M = \{m \in M \mid m_{(-1)} \otimes m_{(0)} = 1 \otimes m\}$.

5.5. THEOREM. *Let H be a Hopf algebra. Assume $\mathscr{C}$ has equalizers. Then*

$$\begin{aligned} {}^{H}_{H}\mathscr{M} &\cong \mathscr{C} \\ H \otimes V &\leftarrow V \\ M &\mapsto {}^{\mathrm{co}\,H}M \end{aligned}$$

are quasi-inverse category equivalences.

From [8] we only recall the main ingredient of the proof. If $M = H \otimes V$, there is the projection $\varepsilon \otimes V \colon M \to V$, which is linear when we endow V with the trivial module structure. In the general case there is a unique H-linear projection p of M onto the subobject ${}^{\mathrm{co}\,H}M$; it is given as the composition

$$M \xrightarrow{\delta} H \otimes M \xrightarrow{S \otimes M} H \otimes M \xrightarrow{\mu} M,$$

that is, by $p(m) := S(m_{(-1)})m_{(0)}$. The diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "p"'] & H \otimes {}^{\mathrm{co}\,H}M \arrow[d, "\varepsilon \otimes \mathrm{id}"] \\ & {}^{\mathrm{co}\,H}M \end{tikzcd}$$

commutes, where the horizontal arrow is one of the isomorphisms that have to be constructed for the proof of 5.5.

Of course the morphism $\eta \otimes V \colon H \otimes V \to M$ corresponds in the general case to the inclusion ${}^{\mathrm{co}\,H}M \to M$.

5.6. COROLLARY. *Let H be a Hopf algebra. Assume $\mathscr{C}$ has equalizers. Then the categories $({}_H^H\mathscr{M}_H, \otimes_H)$ and $({}^H\mathscr{M}_H^H, \square_H)$ are monoidal.*

Proof. We prove the first assertion only. Let N be a left Hopf module and let M be any right H-module. Then the tensor product $M \otimes_H N$ is a split coequalizer since $N \cong \mathrm{H} \otimes {}^{\mathrm{co}\,H}N$ by 5.5. In particular, the tensor product is preserved by $\otimes$ and we are done by 3.3 ∎

Combined with 5.1, 5.3, and 5.4, Theorem 5.5 yields:

5.7. THEOREM. *Let H be a Hopf algebra. Assume $\mathscr{C}$ has equalizers. Then the equivalence*

$$\begin{gathered} {}_H^H\mathscr{M} \cong \mathscr{C} \\ H \otimes V \mapsto V \\ M \mapsto {}^{\mathrm{co}\,H}M \end{gathered}$$

induces equivalences of monoidal categories between

(1) *the category ${}_H^H\mathscr{M}_H$ of two-sided Hopf modules with tensor product $\otimes_H$ and the category of right H-modules,*

(2) *the category ${}_H^H\mathscr{M}^H$ of two-cosided Hopf modules with tensor product $\square_H$ and the category of right H-comodules,*

(3) *the category ${}_H^H\mathscr{M}_H^H$ of two-sided two-cosided Hopf modules with either $\otimes_H$ or $\square_H$ as tensor product, and the category of right Yetter–Drinfel'd H-modules.*

The right module (right comodule) structure on $H \otimes V$ for V a right module (right comodule) is given as in 5.1 and 5.3. The right comodule structure on ${}^{\mathrm{co}\,H}M$ for $M \in {}_H^H\mathscr{M}^H$ is that of ${}^{\mathrm{co}\,H}M$ as a right subcomodule of M. The right module structure on ${}^{\mathrm{co}\,H}M$ for $M \in {}_H^H\mathscr{M}^H$ is given by ${}^{\mathrm{co}\,H}M \otimes H \to M \otimes H \xrightarrow{\mu} M \xrightarrow{p} {}^{\mathrm{co}\,H}M$, that is, $m \leftharpoonup h = S(h_{(1)})mh_{(2)}$ for $m \in {}^{\mathrm{co}\,H}M$ and $h \in H$.

Proof. It remains to check the assertion that we have *monoidal* equivalences. To do this, it is enough to prove that one of the quasi-inverse equivalences is a monoidal functor in each case.

For (1) we show that the isomorphism

$$\begin{gathered} \xi\colon H \otimes V \otimes W \to (H \otimes V) \otimes_H (H \otimes W) \\ h \otimes v \otimes w \mapsto h \otimes v \otimes 1 \otimes w \\ (g \otimes v)h \otimes w \leftarrow g \otimes v \otimes h \otimes w \end{gathered}$$

is a morphism of two-sided Hopf modules. Left linearity and colinearity are trivial. For right linearity, we have

$$\begin{aligned}\xi(g \otimes v \otimes w)h &= (g \otimes v \otimes 1 \otimes w)h \\ &= g \otimes v \otimes (1 \otimes w)h \\ &= g \otimes v \otimes h_{(1)} \otimes w \leftharpoonup h_{(2)} \\ &= (g \otimes v)h_{(1)} \otimes 1 \otimes w \leftharpoonup h_{(2)} \\ &= gh_{(1)} \otimes v \leftharpoonup h_{(2)} \otimes 1 \otimes w \leftharpoonup h_{(3)} \\ &= \xi((g \otimes v \otimes w)h).\end{aligned}$$

Part (2) is formally dual to (1). We deal only with the half of (3) involving $\otimes_H$ since the other half is dual to this. All that remains to check is that ξ is also right colinear; denote the right comodule structure on $H \otimes V \otimes W$ by δ_r.

$$\begin{aligned}\delta_r \xi(h \otimes v \otimes w) &= \delta_r(h \otimes v \otimes 1 \otimes w) \\ &= h_{(1)} \otimes v_{(0)} \otimes 1 \otimes w_{(0)} \otimes h_{(2)} v_{(1)} w_{(1)} \\ &= \xi(h_{(1)} \otimes v_{(0)} \otimes w_{(0)}) \otimes h_{(2)} v_{(1)} w_{(1)} \\ &= (\xi \otimes H)\delta_r(h \otimes v \otimes w).\end{aligned}$$

The coherence condition on monoidal functors follows from the fact that both ways around the rectangle

$$\begin{tikzcd} H \otimes V \otimes W \otimes X \arrow[r, "\xi"] \arrow[d, "\xi"'] & (H \otimes V \otimes W) \otimes_H (H \otimes X) \arrow[d, "\xi \otimes_H \mathrm{id}"] \\ (H \otimes V) \otimes_H (H \otimes W \otimes X) \arrow[r, "\mathrm{id} \otimes_H \xi"'] & (H \otimes V) \otimes_H (H \otimes W) \otimes_H (H \otimes X) \end{tikzcd}$$

are given by $h \otimes v \otimes w \otimes x \mapsto h \otimes v \otimes 1 \otimes w \otimes 1 \otimes x$. ∎

The following example is relevant to the theory of differential calculi over quantum groups. The first order differential calculi over a Hopf algebra H over a commutative ring k studied in [10] consist of a derivation of H to a bimodule $\Omega^1(H)$ replacing the module of differential one forms of the classical situation, where H is of course related to a Lie group. In the classical case the left and right actions of the group on itself induce actions on the differential one forms. In the quantum group situation $\Omega^1(H)$ is a two-sided two-cosided Hopf module (a bicovariant bimodule in the terminology of [10]). The universal first order differential calculus on H is given by the kernel of the multiplication $H \otimes H \to H$

(which is a two-sided two-cosided Hopf module via the left and right actions induced from the left and right tensorand, respectively, and the codiagonal left and right coactions). Denote by H^+ the kernel of ε: $H \to k$.

5.8. EXAMPLE. *Let H be a Hopf algebra over a commutative ring k.*

$$^{\mathrm{co}\, H}(H \otimes H) \ni g \otimes h \mapsto \varepsilon(g)h \in H$$

is an isomorphism of Yetter–Drinfel'd modules, where H is a right module via multiplication and a right comodule via the adjoint coaction. It induces an isomorphism $^{\mathrm{co}\, H}Ker(\nabla_H) \to H^+$, where H^+ is a submodule and a subcomodule of H^{ad}.

The Yetter–Drinfel'd module H with right multiplication and adjoint coaction is known as a special case of results in [2, Sect. 8]. Once we have established the linear and colinear isomorphism above, it follows of course without calculation that it is a Yetter–Drinfel'd module.

Proof. There is a well-known isomorphism

$$\kappa\colon H \otimes H \ni x \otimes y \mapsto xy_{(1)} \otimes y_{(2)} \in H \otimes H.$$

We claim that $\kappa\colon H \otimes H \to H \otimes H$ is linear and colinear on both sides if we endow the domain and codomain with the indicated structures. That is, the domain carries the codiagonal comodule structures and the left and right module structures induced by those of the corresponding tensorands. The left module and comodule structures of the codomain are induced by those of the left tensorand, the right module structure is the diagonal one, and the right comodule structure is the codiagonal structure composed of the canonical comodule structure on the left and the coadjoint coaction on the right tensorand.

The least obvious part is right colinearity (denote the right comodule structures by δ_r):

$$\begin{aligned}\delta_r(\kappa(x \otimes y)) &= \delta_r(xy_{(1)} \otimes y_{(2)})\\ &= x_{(1)}y_{(1)} \otimes y_{(4)} \otimes x_{(2)}y_{(2)}S(y_{(3)})y_{(5)}\\ &= x_{(1)}y_{(1)} \otimes y_{(2)} \otimes x_{(2)}y_{(3)}\\ &= (\kappa \otimes \mathrm{id})\delta(x \otimes y).\end{aligned}$$

The result now follows from 5.7 and the observation that $(\mathrm{id} \otimes \varepsilon)\kappa = \nabla$ and thus $\kappa(\mathrm{Ker}(\nabla)) = H \otimes H^+$, and that $(\varepsilon \otimes H)\kappa(g \otimes h) = \varepsilon(g)h$. ∎

6. Some Consequences

6.1. Corollary. *Let H be a Hopf algebra and assume that $\mathscr{C}$ has equalizers. Then the identity functor is a monoidal equivalence*

$$(\mathscr{I}d, \zeta) : ({}^H_H\mathscr{M}^H_H, \Box_H) \to ({}^H_H\mathscr{M}^H_H, \otimes_H),$$

where the isomorphisms $\zeta_{MN}: M \otimes_H N \to M \Box_H N$ *satisfy* $\zeta(m \otimes n) = m_{(0)}n_{(-1)} \otimes m_{(1)}n_{(0)}$.

Proof. The identity functor is isomorphic to the composition

$$({}^H_H\mathscr{M}^H_H, \Box_H) \xrightarrow{\mathrm{co}\,H(-)} (\mathscr{Y}\mathscr{D}^H_H, \otimes) \xrightarrow{H\otimes(-)} ({}^H_H\mathscr{M}^H_H, \otimes_H)$$

of two monoidal equivalences. It only remains to prove that the induced structure of monoidal functor on the identity has the form we have claimed. It is sufficient to consider the case $M = H \otimes V$ and $N = H \otimes W$ with $V, W \in \mathscr{Y}\mathscr{D}^H_H$. Then ζ_{MN} is the composition

$$(H \otimes V) \otimes_H (H \otimes W) \xrightarrow{\xi^{-1}} H \otimes V \otimes W \xrightarrow{\xi'} (H \otimes V) \Box_H (H \otimes W)$$

(where ξ' is dual to ξ), which satisfies

$$\begin{aligned} \xi'\xi^{-1}(g \otimes v \otimes h \otimes w) &= \xi'((g \otimes v)h \otimes w) \\ &= (g \otimes v)_{(0)}h_{(1)} \otimes (g \otimes v)_{(1)}h_{(2)} \otimes w \\ &= (g \otimes v)_{(0)}(h \otimes w)_{(-1)} \otimes (g \otimes v)_{(1)}(h \otimes w)_{(0)}. \quad \blacksquare \end{aligned}$$

6.2. *Remark.* The coherence condition on ζ is commutativity of

$$\begin{array}{ccc} L \otimes_H M \otimes_H N & \xrightarrow{\zeta \otimes_H N} & (L \Box_H M) \otimes_H N \\ {\scriptstyle L \otimes_H \zeta} \downarrow & & \downarrow {\scriptstyle \zeta} \\ L \otimes_H (M \Box_H N) & \xrightarrow[\zeta]{} & L \Box_H M \Box_H N \end{array}$$

If H is a Hopf algebra with bijective antipode, then 5.7(3) shows ${}^H_H\mathscr{M}^H_H$ to be a braided monoidal category with the tensor product $\otimes_H$. We want to derive the explicit form of the braiding, which has also been used in [10].

6.3. THEOREM. *Let H be a Hopf algebra with bijective antipode. Then $({}_H^H\mathcal{M}_H^H, \otimes_H)$ is a braided monoidal category with braiding*

$$\sigma: M \otimes_H N \to N \otimes_H M$$
$$m \otimes n \mapsto m_{(-2)}n_{(0)}S(n_{(1)}) \otimes S(m_{(-1)})m_{(0)}n_{(2)}.$$

The morphism σ of (H, H)-modules is the unique one satisfying $\sigma(m \otimes n) = n \otimes m$ whenever $m \in {}^{\mathrm{co}\,H}M$ and $n \in N^{\mathrm{co}\,H}$.

Proof. It remains to check that the braiding σ induced in ${}_H^H\mathcal{M}_H^H$ via the monoidal equivalence with $\mathcal{YD}_H^H$ has the stated form.

It is sufficient to consider the case of two-sided two-cosided Hopf modules $M = H \otimes V$ and $N = H \otimes W$ with $V, W \in \mathcal{YD}_H^H$. In this case σ is defined by the commutative diagram

$$\begin{tikzcd}
(H \otimes V) \otimes_H (H \otimes W) \arrow[d, "\sigma"'] & H \otimes V \otimes W \arrow[l, "\xi"'] \arrow[d, "H\otimes\sigma"] \\
(H \otimes W) \otimes_H (H \otimes V) & H \otimes W \otimes V \arrow[l, "\xi"]
\end{tikzcd}$$

where the σ used in the label of the right arrow denotes the braiding in $\mathcal{YD}_H^H$. Thus for $g, h \in H$, $v \in V$, and $w \in W$ we have

$$\begin{aligned}
&\sigma(g \otimes v \otimes h \otimes w)\\
&\quad = \xi(H \otimes \sigma)\xi^{-1}(g \otimes v \otimes h \otimes w)\\
&\quad = \xi(H \otimes \sigma)((g \otimes v)h \otimes w)\\
&\quad = \xi(H \otimes \sigma)(gh_{(1)} \otimes v \leftharpoonup h_{(2)} \otimes w)\\
&\quad = \xi(gh_{(1)} \otimes w_{(0)} \otimes v \leftharpoonup h_{(2)}w_{(1)})\\
&\quad = gh_{(1)} \otimes w_{(0)} \otimes 1 \otimes v \leftharpoonup h_{(2)}w_{(1)}\\
&\quad = g_{(1)}h_{(1)} \otimes w_{(0)} \otimes S(h_{(2)}w_{(1)})S(g_{(2)})g_{(3)}h_{(3)}w_{(2)} \otimes v \leftharpoonup h_{(4)}w_{(3)}\\
&\quad = g_{(1)}(h_{(1)} \otimes w_{(0)})S(h_{(2)}w_{(1)}) \otimes S(g_{(2)})(g_{(3)} \otimes v)h_{(3)}w_{(2)}\\
&\quad = (g \otimes v)_{(-2)}(h \otimes w)_{(0)}S((h \otimes w)_{(1)})\\
&\qquad \otimes S((g \otimes v)_{(-1)})(g \otimes v)_{(0)}(h \otimes w)_{(2)}.
\end{aligned}$$

Evidently $\sigma(m \otimes n) = n \otimes m$ if $m \in {}^{\mathrm{co}\,H}M$ and $n \in N^{\mathrm{co}\,H}$. Now since the H-modules M and N are generated by their subobjects of left invariants and (by an obvious variant of 5.5) right invariants, σ is determined uniquely by this condition. ∎

Assume the antipode of H is an isomorphism. Then applying part (3) of 5.7 to the Hopf algebra H^{cop} yields

6.4. COROLLARY. *Let H be a Hopf algebra whose antipode is an isomorphism. Then there is an equivalence of categories*

$$(-)^{\mathrm{co}\,H} : {}_H^H\mathcal{M}_H^H \to {}^H\mathcal{YD}_H,$$

where $M^{co\,H}$ is a quotient right module of M via the projection $p' \colon M \to M^{co\,H}$ defined by $p'(m) = S^{-1}(m_{(1)})m_{(0)}$.

We want to prove some facts about the composed equivalence between left–right and right–right Yetter–Drinfel'd modules.

6.5. COROLLARY. *Let H be a Hopf algebra whose antipode is an isomorphism. Then for $M \in {}_H^H\mathcal{M}_H^H$ we have ${}^{co\,H}M \cong M^{co\,H}$ as right H-modules and the isomorphism is an isomorphism of right H-comodules when we consider $M^{co\,H}$ as a right comodule via S. In particular, the composed equivalence*

$$ {}^H\mathcal{YD}_H \cong {}_H^H\mathcal{M}_H^H \cong \mathcal{YD}_H^H$$

is isomorphic to one of those described in [7, *Prop.* 2].

Proof. First we note that $\phi_0(m) = S(m_{(-1)})m_{(0)}S(m_{(1)})$ defines an isomorphism $\phi_0 \colon M \to M$ with the inverse ψ_0 defined by $\psi_0(m) = S^{-1}(m_{(1)})m_{(0)}S^{-1}(m_{(-1)})$. Indeed

$$\begin{aligned}\phi_0(m)_{(-1)} \otimes \phi_0(m)_{(0)} \otimes \phi_0(m)_{(1)} \\ = S(m_{(2)}) \otimes S(m_{(-1)})m_{(0)}S(m_{(1)}) \otimes S(m_{(-2)})\end{aligned}$$

and thus

$$\psi_0(\phi_0(m)) = m_{(-2)}S(m_{(-1)})m_{(0)}S(m_{(1)})m_{(2)} = m.$$

Now obviously ϕ_0 maps right invariant elements to left invariant ones and vice versa, as well as ψ_0. More precisely, we have $\phi_0 p' = p$, for $\phi_0(p'(m)) = \phi_0(S^{-1}(m_{(1)})m_{(0)}) = S(S^{-1}(m_{(2)})m_{(-1)})S^{-1}(m_{(1)})m_{(0)} = S(m_{(-1)})m_{(0)} = p(m)$. By definition of the module structures on ${}^{co\,H}M$ and $M^{co\,H}$ this also proves the induced morphism $\phi \colon M^{co\,H} \to {}^{co\,H}M$ to be an isomorphism of right modules. For $m \in M^{co\,H}$ we have $\phi(m) = S(m_{(-1)})m_{(0)}$, of course. Finally ϕ is right colinear as stated, since $\phi(m)_{(0)}\phi(m)_{(1)} = S(m_{(-1)})m_{(0)} \otimes S(m_{(-2)}) = \phi(m_{(0)}) \otimes S(m_{(-1)})$ for $m \in M^{co\,H}$. ∎

This corresponds in our formalism to the assertions 2, 3 in [10, Thm. 2.4]. For assume H is a Hopf algebra over the field k with bijective antipode, and assume ${}^{co\,H}M$ is finite dimensional. Let (v_i) be a basis of ${}^{\mathrm{co}\,H}M$ and let $R_i^j \in H$ be the elements introduced at the end of Section 4. Then colinearity of ϕ means that for $w_i = \phi^{-1}(v_i)$ we have $w_{i(-1)} \otimes w_{i(0)} = S^{-1}(R_i^j) \otimes w_j$ and thus $v_i = \phi(w_i) = S(w_{i(-1)})w_{i(0)} = w_j R_i^j$. This is assertion 2 of the theorem cited. Linearity of ϕ is then just assertion 3.

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